

# Sandpile Model for Relaxation in Complex Systems

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The relaxation in complex systems is in general nonexponential. After an initial rapid decay the system relaxes slowly following a long time tail. In the present paper a sandpile modelation of the relaxation in complex systems is analysed. Complexity is introduced by a process of avalanches in the Bethe lattice and a feedback mechanism which leads to slower decay with increasing time. In this way, some features of relaxation in complex systems: long time tails relaxation, aging, and fractal distribution of characteristic times, are obtained by simple computer simulations.

About 150 years ago Weber and Gauss carried out a simple experiment demonstrating that relaxation in complex systems is not exponential. Investigating the contraction of a silk thread they found that it does not contract so quickly but it relaxes slowly following a power law  $t^{-\alpha}$ . This behavior is not particular of mechanical relaxation, it has been also observed in experiments of magnetic relaxation in spin glasses and high-Tc superconductors, transient current measurements in amorphous semiconductors, dielectric relaxation, and more [1]. This phenomenon of slow relaxation following a power law is called a *long time tail* and it seems to be characteristic of the relaxation in complex systems.

Associated to the slow relaxation dynamics aging is observed, which means that the properties of a system depends on its age [2]. For instance, consider a glass quenched at time  $t = 0$  below its glass transition temperature under an external stress and at time  $t = t_w$  the stress is released. If the system were near equilibrium then its response measured at certain time  $t > t_w$  will be a function of the difference  $t - t_w$  and, therefore, normalized responses taken at different initial times  $t_w$  will collapse in a single curve. However, in the glass material it is observed that the relaxation becomes slower for larger  $t_w$ . The aging phenomena is a consequence of the nonequilibrium dynamics and may be also considered a characteristic of the dynamics of complex systems out from equilibrium.

Sandpiles seem to be simplest systems which also lead to complex behavior.. They have been taken as the paradigm of self-organization since the introduction of these ideas by Bak, Tang, and Wiesenfeld [3]. The evolution of a sandpile under an external perturbation has been extensively studied experimentally [4,5], theoreti-

cally [6,7], and by computer simulations [8]. In general these works analyze the sandpile dynamics close to the critical state, i.e. close to the critical angle.

Slow relaxation dynamics have been observed in the process of compactation of sand heaps under vibrations [9,10]. On the other hand, Jaeger *et al* have measured the temporal decay of the slope angle of sandpiles under vibrations with different intensities [4]. They observed that for high vibration intensities the pile angle relaxes logarithmically slowly to a steady state but it relaxes slower for low intensities. In order to explain the slow dynamics they proposed a simple model, very similar to flux creep models for magnetic relaxation in superconductors. In this way they explain the logarithmic decay observed for high vibration intensities but they could not account for the behavior at low intensities. The model proposed by Jaeger *et al* [4] fails at low vibration intensities because it does not take into account cooperative phenomena which leads to formation of avalanches. The avalanche dynamics should be the main contribution to the relaxation of the pile angle at low vibration intensities, close to the static limit.

In the present paper the angle relaxation of a sandpile under vibrations is investigated. Avalanches are modelled by a mean-field approximation represented in a Bethe lattice. Typical characteristics of complex systems: long time tails relaxation, aging and fractal distributions of time constants are obtained by simple numerical simulations. The main goal of this work is to study the sandpile dynamics far from equilibrium.

In a static sandpile sand grains are in equilibrium unless the angle of the pile overcomes its critical value determined by the static friction coefficient. However, the grains in a pile under vibration are vibrating around cer-

tain quasiequilibrium positions but from time to time the amplitude of its vibrations becomes large. These large amplitude motions carry as a consequence a convective motion in the bulk and the relaxation of the pile angle.

At low intensity vibrations large amplitude grain motions are rare events but from time to time a grain may jump from its quasiequilibrium position. If a surface grain jumps it will fall through the slope of the pile until it collides with another sand grains down the slope. After this collision the initial grain may be trapped with those grains or some of them may fall through the slope. If the last possibility happens then each grain "surviving" the collision will fall through the slope as the initial grain did. This image of an avalanche as an initial object that consecutively drags another resembles a branching process for which the Bethe lattice representation seems to be natural [7].

However, representing the avalanche dynamics in a Bethe lattice is a mean field approach to the problem, it neglects correlations between branches. Notwithstanding, if the pile angle is below its critical value the avalanches will be rare events and, in case of occurrence, will be very sparse. Thus, such an approximation will be acceptable for analyzing the long time relaxation of the pile angle below its critical value, which is the subject of this paper. Away from the critical point the mean field approximation works quite well.

Let us represent the avalanche as a cascade in the Bethe lattice as follows. Firstly, we start with a single node, which could represent in this case a grain. In a further step  $F$  will emerge with probability  $p(\theta)$ , depending on the pile angle  $\theta$ . This operation of generation of  $F$  identical particles starting from one is repeated in the next step to each node of the new group, and so on. If the percolation process overcomes a given length (that of the border of the pile) those nodes beyond the limit constitute the avalanche. If it does not, there would be no avalanche since the cascade was stopped before reaching the base of the pile (frustrated avalanche). By avalanche size we take the number of nodes, of the corresponding Bethe lattice, in the last step.

After an avalanche the sandpile *autorganizes itself* with the new number of grains (i.e. a new slope is calculated with the remaining grains). The occurrence of avalanches will carry as consequence a decrease in the number of grains in the pile and, therefore, a decrease of  $p(\theta)$ . As more avalanches take place less probable is the occurrence of a new avalanche. Thus to characterize this feedback mechanism a relation between  $p$  and the number of grains in the pile  $N$  is needed.

The drag probability  $p(\theta)$  is a function of the slope. Its value is determined by the competition of two contrary forces: the gravity which conspires against the stability of the slope and the friction which favours the slope stability. Since the slope forms an angle  $\theta$  with the horizontal plane the component of the gravity force in the slope direction will be larger with the increase of this angle, varying from zero to a maximum value when  $\theta$  goes from

zero to  $\pi/2$ . Therefore it is plausible to assume that the contribution of the gravity and, therefore, the tendency of falling down the slope is proportional to  $\sin\theta$ . On the contrary, the resistance to this tendency given by the static friction decreases with decreasing the pile angle, varying from a maximum value to zero when  $\theta$  goes from zero to  $\pi/2$ . Hence, it is also plausible to assume that the resistance to the falling down is proportional to  $\cos\theta$ .

Thus the slope dependence of  $p$  should be given through the ratio of both tendencies  $\sin\theta/\cos\theta = \tan\theta$ . Based on this hypothesis we propose the exponential relation

$$p(\theta) = \exp(-A/\tan\theta) , \quad (1)$$

where  $A$  is a parameter determined by the gravitational field, the friction, and vibration intensity. Notices that  $p(0) = 0$  and  $p(\pi/2) = 1$ .

On the other hand, as it is well known the number of grains in a pile with slope angle  $\theta$  is given by

$$N = Bx^3 \tan\theta \quad (2)$$

where  $B$  is a geometrical factor and  $x = D/d$  is the ratio between a characteristic size of the pile base  $D$  and sand grain  $d$ . Combining equations 1 and 2 it is obtained

$$p(N) = \exp(-cx^3/N) . \quad (3)$$

where  $c = AB$ . Thus equation 1 relates the dragging probability with the number of grains in the pile.

The parameter  $c$  should not depend on the size of the system. It must be a function of parameters describing the vibrations, amplitude  $a$  and frequency  $\omega$ , and of the gravitational field acceleration  $g$ . The only nondimensional combination of this magnitudes is given by the ratio between the vibration acceleration  $a\omega^2$  and  $g$  and, therefore,  $c = c(a\omega^2/g)$ . This conclusion obtained from dimensional analysis is corroborated by experiments on sandpiles under vibrations [11] which shown that the ratio  $a\omega^2/g$  is the relevant parameter.

For a given value of  $c$  and  $x$  there is a critical number of grains in the pile  $N_c$ . This value can be found recalling that in the Bethe lattice the critical value of  $p$  for percolation exists is  $1/F$ , resulting

$$N_c = cx^3/\ln F . \quad (4)$$

In a static sandpile (i.e. no vibrations) this value corresponds with the number of grains in the pile at the angle of repose [7].

Another magnitude of interest is the penetration length, the number of steps  $n(\theta)$  in the Bethe lattice for an avalanche take place. This magnitude must be proportional to the length of the pile slope and, therefore, should be given by the expression

$$n(\theta) = x/\cos\theta \quad (5)$$

Notice that the possible existence of a geometrical factor in equation 5 may be absorbed in  $x$ , redefining  $B$  in

equation 2. Moreover, the relation between  $n$  and  $N$  may be easily obtained using equation 2.

Equations 3 and 5 relates the parameters of the pile with those of its Bethe lattice representation. They were obtained here in a different way than in [7]. Other dependencies between the dragging probability and the number of grains in the sandpile may be proposed. Notwithstanding, as it is discussed latter, the precise form of this functional dependence is not too relevant.

The numerical experiment of relaxation is performed as follows. We start with a certain number of  $N$ . In a first step, we test if an avalanche takes place using the Bethe lattice representation. If it does then we recalculate the value of  $p(N)$  by simply resting the size of the avalanche to  $N$  and using equation 3. Then this step is repeated again and again.

If avalanches are considered as instantaneous the number of steps is a measure of time. This approximation is valid for low vibrations intensities. In this case grain jumps which trigger avalanches are rare events and, therefore, the time between to successive grain jumps will be much larger than the duration of avalanches.

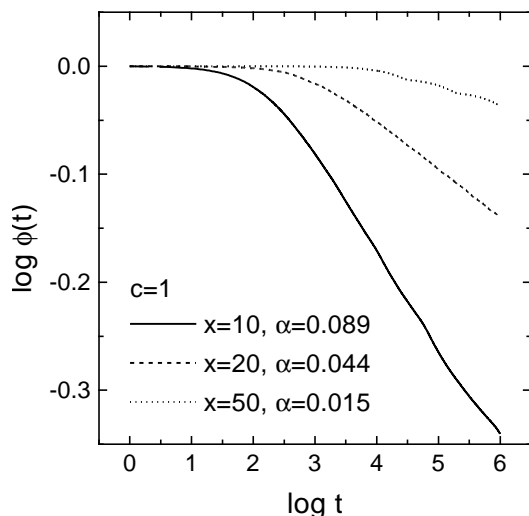


FIG. 1. Normalized relaxation  $\phi(t) = N(t)/N(t=0)$  of the number of grains in the pile for different values of  $x$ . The initial condition is  $N(t=0) = N_c$  in all cases. Insets show the values of  $\alpha$  obtained from the fit to the power tail  $t^{-\alpha}$ .

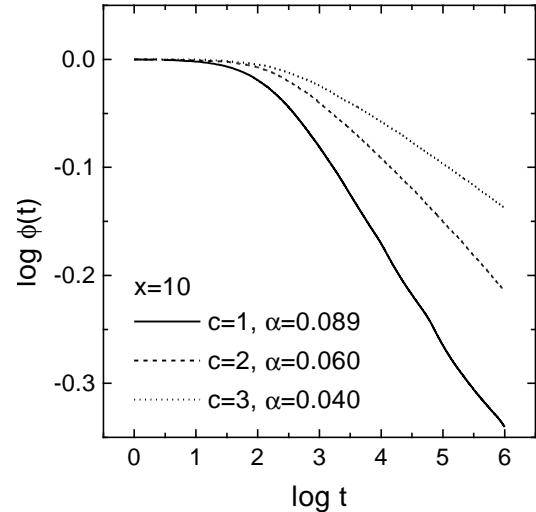


FIG. 2. Normalized relaxation  $\phi(t) = N(t)/N(t=0)$  of the number of grains in the pile for different values of  $c$ . The initial condition is  $N(t=0) = N_c$  in all cases. Insets show the values of  $\alpha$  obtained from the fit to the power tail  $t^{-\alpha}$ .

The normalized decay of  $N$  in time for different values of  $x$  and  $c$  is shown in figures 1 and 2. In all cases we took as initial condition  $N(t=0) = N_c$ , the number of grains at the angle of repose in the corresponding static limit, and we have averaged over 10000 realizations. As it can be seen the relaxation follows a fast decay at short times but it is slower than an exponential at long times. The log-log plot reveals a straight line at long times characteristic of a long time tail  $t^{-\alpha}$ . Insets show the values of  $\alpha$  obtained from the fit to this power tail, having relative errors of the order of 0.01.

The characteristic exponent  $\alpha$  decreases towards zero with increasing  $x$ , i.e. the relaxation is slower for larger sandpiles. It is expected that it is strictly zero for  $x \rightarrow \infty$ , which means that an infinite sandpile will be in equilibrium. This limit may be the equivalent of the ferromagnetic state of magnetic materials where domain are macroscopic structures having finite magnetization. On the other hand, with increasing  $c$ ,  $\alpha$  decreases towards zero and since the relaxation should be slower for smaller vibrations intensities we may conclude that  $c$  must increase with decreasing the vibration intensity. Hence,  $c$  is a monotonically decreasing function of  $a\omega^2/g$ .

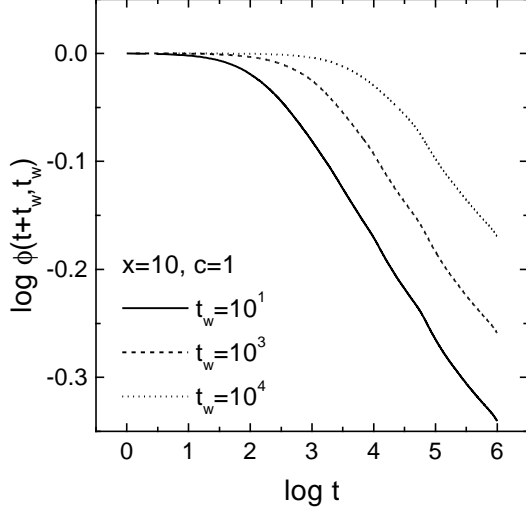


FIG. 3. Aging. Normalized relaxation function  $\phi(t + t_w, t_w) = N(t + t_w)/N(t_w)$  taken at three different initial times  $t_w$ .

The existence of aging in our model is illustrated in figure 3. We have plotted the normalized relaxation at different ages (steps) of the system (simulation) by taking as initial time  $t = t_w$ . As it can be seen the relaxation becomes slower with increasing the age of the system, in agreement with experiments in structural and spin glasses [2]. Thus, since the angle relaxation becomes slower with the age of the pile then it will never reach an equilibrium angle and, therefore, properties like translational invariance and the fluctuation dissipation theorem do not hold [2].

Associated with these slow relaxation dynamics and aging phenomena we expect to observe a wide distribution of time between avalanches. As time increases the time between two consecutive avalanches  $\Delta t$  becomes larger and larger, because after an avalanche the occurrence of a new avalanche becomes smaller. Therefore, it is expected that the mean time between avalanches diverges as  $t \rightarrow \infty$ . Hence, the distribution of time between avalanches  $n(\Delta t)$  should satisfy the asymptotic behavior for large  $\Delta t$

$$n(\Delta t) \sim \Delta t^{-1-\beta} \quad (6)$$

with  $0 < \beta < 1$ .

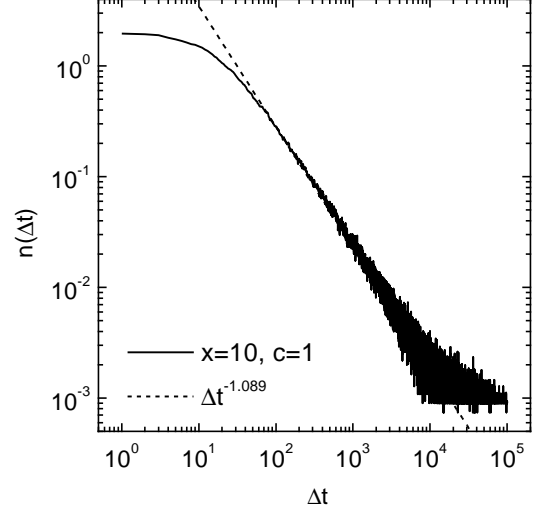


FIG. 4. Distribution of time between avalanches. The dashed line is a power tail with exponent  $-1 - \alpha$ ,  $\alpha$  being the exponent of the long time tail shown in figure 1.

This hypothesis is confirmed in our simulations. Figure 4 shows the distribution of time between avalanches for  $x = 10$  and  $c = 1$ . It is approximately constant for small values and then it decays following a power law in more than two decades. The plateau at small values of  $\Delta t$  is associated to the rapid decay observed at short times while the tail for large  $\Delta t$  should be related to the long time tail. Thus, there should be some connection between the distribution of time between avalanches and the long time relaxation.

In fact, if we assume that the relaxation is given by the superposition of exponential relaxations, the relaxation time being the time between avalanches, the temporal decay of  $N$  will be given by

$$N(t) = N_0 \int_0^\infty d\Delta t \, n(\Delta t) \exp(-t/\Delta t). \quad (7)$$

Then, for long times equation 7 gives the asymptotic behavior

$$N(t) \sim t^{-\beta}. \quad (8)$$

Thus  $\alpha = \beta$ , the exponent of the long time tail and the characteristic exponent of the distribution of time between collisions are mutually determined. This result is corroborated in figure 4 where it can be seen that a power decay  $\sim \Delta t^{-1-\alpha}$  fits quite well the distribution of time between avalanches for large  $\Delta t$ . Long time tails relaxation and fractal distributions of time constants are the same phenomena manifested in different windows [12].

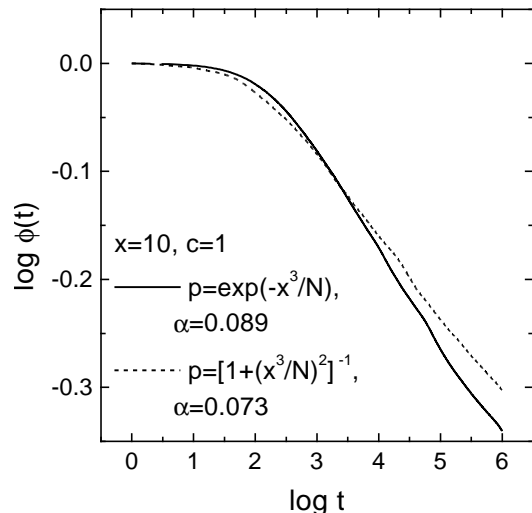


FIG. 5. Normalized relaxation  $\phi(t) = N(t)/N(t=0)$  of the number of grains in the pile for two different functional dependences  $p = p(N)$ . The initial condition is  $N(t=0) = N_c$  in both cases. Insets show the values of  $\alpha$  obtained from the fit to the power tail  $t^{-\alpha}$ .

Finally we want to mention that these simulations were also carried out assuming other functional dependences between the dragging probability and the number of grains in the pile  $p(N)$ . In all cases the results were qualitatively similar to those presented here using equation 3, reflecting that the precise dependence is not determinant. For instance, in figure 5 the normalized relaxation for two different functional forms of  $p(N)$  is plotted. In both cases the long time relaxation follows a long time tail but with different exponents. The qualitative form of the relaxation does not depend on the detailed form of  $p(N)$ , i.e. on the detailed nature of the system. Nevertheless, the form of  $p(N)$  does determine the value of the long time tail exponent  $\alpha$  suggesting that different universality classes are possible.

We conclude that the introduction of a Bethe lattice representation for the avalanches and a feedback mechanism describes quite well the principal features of the relaxation in sandpiles under low intensity vibrations. The

proposed representation leads to long time tails relaxation, aging, and fractal distributions of time constants, which are characteristic properties of the dynamics of complex systems out from the equilibrium. The slow relaxation dynamics and related properties are a consequence of the feedback mechanism, but the detailed nature of this feedback is not relevant for the qualitative behavior.

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